Conclusion

This final problem set wraps up some loose ends and points out a way forward.

1 Partial Derivatives

In **T2**, we mentioned a few more advanced rules for using partial derivatives. In this section we'll consider some problems that require them.

[2] **Problem 1.** For a function $f(x, y)$ with differential

$$df = f_x\, dx + f_y\, dy$$

prove that the identity

$$\left.\frac{\partial f_x}{\partial y}\right|_x = \left.\frac{\partial f_y}{\partial x}\right|_y$$

must be satisfied. Conversely, if you are given a differential that doesn't satisfy this, you can be sure that it's not the differential of *any* function. For an ideal gas, write down the differential $đQ$ in terms of dT and dV, and use it to prove that Q is not a state function.

Solution. The desired result follows immediately from the fact that partial derivatives commute. Next, for an ideal gas we have

$$đQ = C_V\, dT + \left.\frac{\partial Q}{\partial V}\right|_T dV.$$

For an ideal gas only, the amount of heat that must be supplied as the volume changes, to keep the temperature constant, is just the opposite of the work done. Thus

$$đQ = C_V\, dT + P\, dV, \quad \left.\frac{\partial C_V}{\partial V}\right|_T = 0, \quad \left.\frac{\partial P}{\partial T}\right|_V = \frac{nR}{V}.$$

Hence the identity is not satisfied and heat is not a state function.

[2] **Problem 2.** Starting from the differential $dU = -P\, dV + T\, dS$, show that

$$-\left.\frac{\partial P}{\partial S}\right|_V = \left.\frac{\partial T}{\partial V}\right|_S.$$

This is an example of a Maxwell relation.

Solution. This follows immediately from the result in problem 1, since U is a state function. The same idea holds for any state function, including the enthalpy, Helmholtz free energy, and Gibbs free energy, giving rise to several more Maxwell relations.

[3] **Problem 3** (Blundell 12.2)**.** Consider a gas with the equation of state $PV = f(T)$. Show that

$$\left.\frac{\partial P}{\partial T}\right|_V = \frac{1}{V}\frac{df}{dT}, \quad \left.\frac{\partial V}{\partial T}\right|_p = \frac{1}{p}\frac{df}{dT}.$$

Also show that

$$\left.\frac{\partial Q}{\partial V}\right|_P = C_P \left.\frac{\partial T}{\partial V}\right|_P, \quad \left.\frac{\partial Q}{\partial P}\right|_V = C_V \left.\frac{\partial T}{\partial P}\right|_V.$$

In an adiabatic process, we have $đQ = 0$. By taking the differential with respect to dp and dV, show that PV^γ is constant, thus generalizing the rule for adiabatic processes for ideal gases.

Solution. The first two follow by differentiating both sides of $pV = f(T)$ while holding either V or p constant. The next two follow by using the chain rule (replace the heat capacities with their definitions). We see that

$$đQ = \left.\frac{\partial Q}{\partial P}\right|_V dp + \left.\frac{\partial Q}{\partial V}\right|_p dV,$$

so

$$C_V V \frac{df}{dT} dp + C_p p \frac{df}{dT} = 0,$$

so

$$C_V (dp/p) + C_p (dV/V) = 0.$$

Integrating, we see that $C_V \log p + C_p \log V = \text{const}$, so $pV^\gamma = \text{const}$.

[3] **Problem 4** (Kardar)**.** We know from kinetic theory and the equipartition theorem that the energy of an ideal gas depends only on the temperature. However, this can also be deduced from pure thermodynamics. Starting from $dE = T\,dS - P\,dV$, show that $PV = Nk_BT$ implies that E can only depend on T. You will have to use a Maxwell relation.

Solution. For this kind of setup, every quantity can be written in terms of two independent variables; in particular, we can always write E in terms of T and S. Therefore, the desired conclusion follows if we can show that $(\partial E/\partial S)|_T = 0$, so that the dependence on S drops out.

The key Maxwell relation here is

$$\left.\frac{\partial V}{\partial S}\right|_T = \left.\frac{\partial T}{\partial p}\right|_V,$$

which can be derived by looking at the Helmholtz free energy $dF = -S\,dT - p\,dV$. Now, starting from $dE = T\,dS - p\,dV$, we have

$$\left.\frac{\partial E}{\partial S}\right|_T = T - p\left.\frac{\partial V}{\partial S}\right|_T = T - p\left.\frac{\partial T}{\partial p}\right|_V.$$

From the ideal gas law, the second term is $-p\frac{V}{Nk_B} = -T$, giving the result.

[3] **Problem 5.** INPhO 2020, problem 1.

Solution. See the official solutions here.

2 Extra Problems

In **W1** through **W3**, we studied waves that were described by linear equations. There are many subtle new features that appear when we allow for nonlinearity. Unfortunately, they're so subtle that the vast majority of university physics degrees never cover *any* of this subject. (For example, at my institution, Stanford, there aren't any physics courses that cover nonlinear waves even at the graduate level; you have to go to other departments to find them.) So if you solve the below problems, you'll know more about them than almost all physicists!

[5] **Problem 6.** GPhO 2017, problem 3. A pedagogical question on shock waves, a nonlinear wave effect. For background, see chapter 32 of Blundell.

Solution. See the official solutions here.

[5] **Problem 7.** IPhO 2008, problem 2. On Cherenkov radiation, a famous shock wave effect.

The following questions contain applied physics topics. They were not included in the main problem sets mostly because they used a variety of topics about equally, but they're great review practice.

[5] **Problem 8.** IPhO 2013, problem 3. A solid question on an ice sheet, using thermo and data analysis. Use the data sheet of physical constants provided in the folder.

[5] **Problem 9.** IZhO 2022, problem 2. A practical problem on estimating the greenhouse effect.

Solution. See the official solutions here.

[5] **Problem 10.** IPhO 2017, problem 2. Analyzing extreme weather events with fluids and waves.

[5] **Problem 11.** IPhO 2018, problem 3. Some models in biophysics, using fluids and thermo.

[5] **Problem 12.** IPhO 2021, problem 1. The basics of geophysics, using fluids and waves.

[5] **Problem 13.** APhO 2014, problem 1. A very hard problem on atmospheric convection.

[5] **Problem 14.** APhO 2012, problem 3. A somewhat technical problem that combines polarization and interference, to derive a neat effect called a "geometric phase".

3 Extra Olympiads

If you've made it this far, you're probably training for the IPhO. I've reserved questions from a few Olympiads you can take as full-length exams for practice. Note that NBPhO exams are usually 2 days long, with 5 hours per day. However, they include experimental questions that you can't do at home, and they're somewhat easier than the IPhO, APhO, or EuPhO, so I've adjusted the time limits accordingly.

[12] **Problem 15.** GPhO 2019.

[12] **Problem 16.** NBPhO 2019.

[12] **Problem 17.** NBPhO 2021.

[15] **Problem 18.** IPhO 2015.

[15] **Problem 19.** NBPhO 2022.

[15] **Problem 20.** APhO 2018.

[18] **Problem 21.** EuPhO 2020.

[18] **Problem 22.** APhO 2017.

[20] **Problem 23.** IPhO 2019.

[20] **Problem 24.** EuPhO 2021.

[20] **Problem 25.** APhO 2019.

Once you finish this and get your IPhO medal, you've outgrown Olympiad physics. The stuff covered in Olympiad problems *is* real physics – the rough estimates and calculations a physicist would do when exploring a new idea actually resemble what you see in IPhO and APhO problems. But the IPhO and APhO are restricted by a syllabus that assumes very little mathematical background. There is a whole lot more beautiful, fascinating stuff you can do once you pick up vector calculus and linear algebra, the two most important pieces of the physicist's toolkit. For more advice on what you can do next, see my second advice file. You have a lot of options, and you've earned it.